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On representations of locally inverse $*$-semigroups[1]

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Abstract

The purpose of this paper is to obtain an analogous representation of the Preston-Vagner Representation for locally inverse $*$-semigroups which is a generalization of [7].

Firstly, by introducing a concept of a *π-set* (which is slightly different from the one in [7]), we shall construct *the π-symmetric locally inverse $*$-semigroup* $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ on a π-set $X(\pi';\omega;\{\sigma_{e,f}\})$, and show that $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ is a locally inverse $*$-semigroup and that any locally inverse $*$-semigroup can be embedded up to $*$-isomorphism in $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ on a π-set $X(\pi';\omega;\{\sigma_{e,f}\})$. Moreover, we shall show that the wreath product (in the sense of Cowan[1]) of locally inverse $*$-semigroups is also a locally inverse $*$-semigroup.

1 Introduction

A semigroup S with a unary operation $* : S \to S$ is called a *regular $*$-semigroup* if it satisfies

(i) $\quad (x^*)^* = x,$

(ii) $\quad (xy)^* = y^*x^*,$

(iii) $\quad xx^*x = x.$

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if it satisfies $e^* = e$. For any subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively. The following result is well-known, and we use it frequently throughout this paper.

[1]This is the abstract and the details will be published elsewhere. The results of § 2 and 4 were obtained after the conference.

Result 1.1 (see [4]). *Let S be a regular $*$-semigroup. Then we have the followings:*

(1) $E(S) = P(S)^2$, *more precisely, for any $e \in E(S)$, there exist $f, g \in P(S)$ such that $f\mathcal{R}e\mathcal{L}g$ and $e = fg$;*

(2) *for any $a \in S$ and $e \in P(S), a^*ea \in P(S)$;*

(3) *each $\mathcal{L}$-class and each $\mathcal{R}$-class have one and only one projection.*

A regular $*$-semigroup S is called a *locally inverse $*$-semigroup* if, for any $e \in E(S)$, eSe is an inverse subsemigroup of S.

Lemma 1.2 *A regular $*$-semigroup S is a locally inverse $*$-semigroup if and only if, for each $e \in P(S), eSe$ is an inverse subsemigroup of S.*

A regular $*$-semigroup S is called a *generalized inverse $*$-semigroup* if $E(S)$ forms a normal band, that is, $E(S)$ satisfies the identity $xyzx = xzyx$. It is obvious that a generalized inverse $*$-semigroup is a locally inverse $*$-semigroup.

Remark. It is clear that a regular $*$-semigroup S is a generalized inverse $*$-semigroup if and only if it satisfies the following condition:

$$\text{for any } e, f, g, h \in P(S),\ efgh = egfh \text{ (in } S).$$

However, we remark that even if a locally inverse $*$-semigroup S satisfies the condition

$$\text{for any } e, f, g \in P(S),\ efge = egfe \text{ (in } S),$$

it is not always a generalized inverse $*$-semigroup. The second remark of [6] is its counterexample.

Let X be a set. If $X = \bigcup\{X_i : i \in I\}$ is a partition of X, denote it by $X = \sum\{X_i : i \in I\}$. For a mapping $\alpha : A \to B$, denote the domain and the range of α by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset C of A, $\alpha|_C$ means the restriction of α to C.

Let $\mathcal{I}_X$ be the symmetric inverse semigroup on a set X. For a subset A of X, 1_A means the identity mapping on A. Let $\mathcal{A}$ be an inverse subsemigroup of $\mathcal{I}_X$ and $\theta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ a mapping. Denote the image $(\alpha, \beta)\theta$ of an ordered pair (α, β) by $\theta_{\alpha,\beta}$. Set $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{A}\}$. If $\mathcal{M}$ satisfies the following conditions:

(C1) $\theta_{\alpha,\beta}^{-1} = \theta_{\beta^{-1},\alpha^{-1}}$,

(C2) $\theta_{\alpha,\alpha^{-1}} = 1_{r(\alpha)}$,

(C3) $\theta_{1_{d(\alpha)},\alpha} = 1_{d(\alpha)}$,

(C4) $\theta_{\alpha,\beta}\beta\theta_{\alpha\theta_{\alpha,\beta}\beta,\gamma} = \theta_{\alpha,\beta\theta_{\beta,\gamma}\gamma}\beta\theta_{\beta,\gamma}$,

we call it *the structure sandwich set* of $\mathcal{A}$ determined by θ.

Result 1.3 (see [7]) *Let $\mathcal{A}$ be an inverse subsemigroup of the symmetric inverse semigroup $\mathcal{I}_X$ on a set X, and $\mathcal{M}$ the structure sandwich set of $\mathcal{A}$ determined by a mapping $\theta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{A}$ as follows:*

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$
$$\alpha^* = \alpha^{-1}$$

Then $\mathcal{A}(\circ, *)$ becomes a regular $*$-semigroup.

Hereafter, we call such a semigroup $\mathcal{A}(\circ, *)$ a *regular $*$-semigroup of partial one-to-one mappings* determined by the *structure sandwich set* $\mathcal{M}$, and denote it by $\mathcal{A}(\mathcal{M})$. The notation and terminology are those of [3] and [4] , unless otherwise stated.

In § 2, we shall firstly consider a *π-set* $X(\pi'; \omega; \{\sigma_{e,f}\})$ which is a set X with a partition $\pi' : X = \sum\{X_e : e \in \Lambda\}$, a reflexive and symmetric relation ω on Λ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, where $\sigma_{e,f}$ is a bijection of X_e onto X_f. We remark that a *π-set*, defined in this paper, is slightly different from the one in [7], which is called a *strong π-set* in this paper. The set $\mathcal{LI}_{X(\pi')}$, say, of all partial one-to-one π-mappings on $X(\pi'; \omega; \{\sigma_{e,f}\})$ is an inverse subsemigroup of $\mathcal{I}_X$. By using $\{\sigma_{e,f} : (e, f) \in \omega\}$, we shall construct a structure sandwich set $\mathcal{M}$, and show that $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ is a locally inverse $*$-semigroup. We call such a semigroup $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ *the π-symmetric locally inverse $*$-semigroup* on a π-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $\mathcal{M}$.

In § 3, we shall show that any locally inverse $*$-semigroup is embedded up to $*$-isomorphism in the π-symmetric locally inverse $*$-semigroup $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ on a π-set $X(\pi'; \omega; \{\sigma_{e,f}\})$.

As a generalization of [2], Cowan [1] gave us the definition of the wreath product $S\mathrm{wr}T(X)$ of inverse semigroups S and $T(X)$, where $T(X)$ is an inverse subsemigroup of $\mathcal{I}_X$. And he showed that the wreath product $S\mathrm{wr}T(X)$ is also an inverse semigroup. In § 4, we shall show that the wreath product of locally inverse $*$-semigroups S and $T(X)$ $(\subseteq \mathcal{LI}_{X(\pi')})$ is a locally inverse $*$-semigroup. Moreover, we shall obtain that the wreath product of generalized inverse $*$-semigroups is also a generalized inverse $*$-semigroup.

2 π-Symmetric locally inverse $*$-semigroups

Let X be a non-empty set. If there exist a partition $X = \sum\{X_e : e \in \Lambda\}$ and a reflexive and symmetric relation ω on Λ such that

(i) for each $(e, f) \in \omega$, there exists a bijection $\sigma_{e,f} : X_e \to X_f$,

(ii) for all $e \in \Lambda$, $\sigma_{e,e} = 1_{X_e}$,

(iii) for any $(e, f) \in \omega$, $\sigma_{f,e} = \sigma_{e,f}^{-1}$, then X is called a *π-set* with a partition $\pi' : X = \sum\{X_e : e \in \Lambda\}$, a relation ω and a set of mappings $\{\sigma_{e,f} : (e,f) \in \omega\}$, and denote it by $X(\pi';\omega;\{\sigma_{e,f}\})$, or simply by $X(\pi')$. If a π-set $X(\pi';\omega;\{\sigma_{e,f}\})$ satisfies the following two conditions

(iv) ω is transitive, that is, it is an equivalence relation,

(v) for $(e,f),(f,g) \in \omega$, $\sigma_{e,f}\sigma_{f,g} = \sigma_{e,g}$,

it is called a *strong π-set.*

Let $X(\pi';\omega;\{\sigma_{e,f}\})$ be a π-set. A subset A of X is called a *π-single subset* of X if for each $e \in \Lambda$, there exists at most one element $f \in \Lambda$ such that $X_f \cap A \neq \Box$ and $(e,f) \in \omega$. We consider the empty set as a π-single subset. Denote the family of all π-single subsets of $X(\pi';\omega;\sigma_{e,f}\})$ by $\mathbf{T}$.

A mapping α in the symmetric inverse semigroup $\mathcal{I}_X$ on X is called *a partial one-to-one π-mapping* of $X(\pi';\omega;\{\sigma_{e,f}\})$ if $d(\alpha)$ and $r(\alpha)$ are π-single subsets. Let $\mathcal{LI}_{X(\pi')}$ be the set of all partial one-to-one π-mappings of $X(\pi';\omega;\{\sigma_{e,f}\})$, that is, $\mathcal{LI}_{X(\pi')} = \{\alpha \in \mathcal{I}_X : d(\alpha), r(\alpha) \in \mathbf{T}\}$. The following lemma is clear.

Lemma 2.1 *The set $\mathcal{LI}_{X(\pi')}$, defined above, is an inverse subsemigroup of $\mathcal{I}_X$.*

For $A, B \in \mathbf{T}$, define a mapping $\theta_{A,B}$ as follows:

$$
\begin{aligned}
&d(\theta_{A,B}) = \{x \in A : \text{there exist } e,f \in \Lambda \text{ such that } (e,f) \in \omega, \\
&\qquad\qquad\qquad\qquad\qquad\qquad x \in X_e \text{ and } x\sigma_{e,f} \in B\}, \\
(3.1)\qquad &r(\theta_{A,B}) = \{y \in B : \text{there exist } e,f \in \Lambda \text{ such that } (e,f) \in \omega, \\
&\qquad\qquad\qquad\qquad\qquad\qquad y \in X_f \text{ and } y\sigma_{f,e} \in A\}, \\
&x\theta_{A,B} = x\sigma_{e,f} \qquad (x \in d(\theta_{A,B}) \cap X_e, (e,f) \in \omega).
\end{aligned}
$$

For any $\alpha, \beta \in \mathcal{LI}_{X(\pi')}$, define $\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}$. Since a subset of π-single subset is also a π-single subset, we have that $\theta_{\alpha,\beta} \in \mathcal{LI}_{X(\pi')}$ for all $\alpha, \beta \in \mathcal{LI}_{X(\pi')}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{LI}_{X(\pi')}\}$.

Lemma 2.2 *The set $\mathcal{M}$, defined above, is the structure sandwich set of $\mathcal{LI}_{X(\pi')}$ determined by a mapping $\theta : \mathcal{LI}_{X(\pi')} \times \mathcal{LI}_{X(\pi')} \to \mathcal{LI}_{X(\pi')}$ $((\alpha,\beta) \mapsto \theta_{\alpha,\beta})$. Therefore, $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ is a regular $*$-semigroup.*

We call the set $\mathcal{M}$, defined above, *the structure sandwich set determined by a π-set* $X(\pi';\omega;\{\sigma_{e,f}\})$.

It is clear that each projection of $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ is the identity mapping 1_A on a π-single subset A. Let 1_A be any projection and let α, β be any projections of $1_A \circ \mathcal{LI}_{X(\pi')} \circ 1_A$. There exist $B, C \in \mathbf{T}$ such that $\alpha = 1_A \circ 1_B \circ 1_A$ and $\beta = 1_A \circ 1_C \circ 1_A$. Then $\alpha = \theta_{A,B}\theta_{A,B}^{-1} = 1_{d(\theta_{A,B})}$ and $\beta = 1_{d(\theta_{A,C})}$. Since $d(\theta_{A,B}) \subseteq A$ and $d(\theta_{A,C}) \subseteq A$, $\theta_{1_{\theta_{A,B}}, 1_{\theta_{A,C}}} \subseteq \theta_{A,A} = 1_A$.

Similarly $\theta_{1_{\theta_{A,C}},1_{\theta_{A,B}}} \subseteq 1_A$. Then $\alpha\circ\beta = \beta\circ\alpha$. Therefore, $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ ia a locally inverse $*$-semigroup. We call it *the π-symmetric locally inverse $*$-semigroup* on $X(\pi';\omega;\{\sigma_{e,f}\})$ with the structure sandwich set $\mathcal{M}$. Now, we have the following theorem.

Theorem 2.3 *Let X be a π-set with a partition $\pi' : X = \sum\{X_e : e \in \Lambda\}$, a relation ω on Λ and a set of mappings $\{\sigma_{e,f} : (e,f) \in \omega\}$, and let $\mathcal{M}$ be the structure sandwich set determined by $X(\pi';\omega;\{\sigma_{e,f}\})$. Then $\mathcal{LI}_{X(\pi')}$ is an inverse subsemigroup of $\mathcal{I}_X$. Moreover, $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ is a locally inverse $*$-semigroup.*

Let $X(\pi';\omega;\{\sigma_{e,f}\})$ be a strong π-set, where $\pi' = \sum\{X_e : e \in \Lambda\}$. Since ω is an equivalence relation on Λ, there exists the partition $\Lambda = \sum\{\Lambda_i : i \in I\}$ induced by ω. For each $i \in I$, denote the subset $\cup\{X_e : e \in \Lambda_i\}$ by $\mathbf{X}_i$.

Lemma 2.4 *A subset A of X is a π-single subset if and only if it satisfies the condition that for any $i \in I$, $A \cap \mathbf{X}_i \neq \Box$ impies $A \cap \mathbf{X}_i \subseteq Xe$ for some $e \in \Lambda_i$.*

Let $\mathcal{M}$ be the structure sandwich set determined by $X(\pi';\omega;\{\sigma_{e,f}\})$. By Theorem 3.6 of [7] , $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ is a generalized inverse $*$-semigroup. We call such a semigroup *the π-symmetric generalized inverse $*$-semigroup* and denote it by $\mathcal{GI}_{X(\pi')}(\mathcal{M})$ instead of $\mathcal{LI}_{X(\pi')}(\mathcal{M})$.

Corollary 2.5 (Theorem 3.6 [7]) *Let X be a strong π-set with a partition $\pi' : X = \sum\{X_e : e \in \Lambda\}$, an equivalence relation ω on Λ and a set of mappings $\{\sigma_{e,f} : (e,f) \in \omega\}$, and let $\mathcal{M}$ be the structure sandwich set determined by $X(\pi';\omega)$. Then $\mathcal{GI}_{X(\pi')}(\mathcal{M})$ is a generalized inverse $*$-semigroup.*

3 Representations

Let S be a locally inverse $*$-semigroup and $\mathcal{I}_S$ the symmetric inverse semigroup on S. In this section, denote $E(S)$ and $P(S)$ simply by E and P, respectively. Since each $\mathcal{L}$-class has one and only one projection, $\pi' : S = \sum\{L_e : e \in P\}$ is a partition of S, where L_e denotes the $\mathcal{L}$-class containing e. Let $\omega = \{(e,f) \in P \times P : e\mathcal{R}g\mathcal{L}f$ for some $g \in E\}$. It is clear that ω is a reflexive and symmetric relation on P. For $(e,f) \in \omega$, define $\sigma_{e,f} : L_e \to L_f$ by $x\sigma_{e,f} = xf$. It follows from Green's Lemma that $S(\pi';\omega;\{\sigma_{e,f}\})$ is a π-set. Let $\mathbf{T}$ be the set of all π-single subsets of $S(\pi';\omega;\{\sigma_{e,f}\})$ and $\mathcal{M}$ the structure sandwich set determined by $S(\pi';\omega;\{\sigma_{e,f}\})$. By Theorem 2.5, $\mathcal{LI}_{S(\pi')}(\mathcal{M})$ is a locally inverse $*$-semigroup.

For any $a \in S$, let $\rho_a : Sa^* \to Sa$ be a mapping defined by

$$x\rho_a = xa.$$

It is trivial that ρ_a and ρ_{a^*} are mutually inverse mappings of Sa^* and Sa onto each other, and hence $\rho_a \in \mathcal{I}_S$. A subset of S is said to be *$\mathcal{L}$-full* if it is a union of some $\mathcal{L}$-classes of S.

Lemma 3.1 (1) *For any $a \in S$, $\rho_a \in \mathcal{LI}_{S(\pi')}$.*

(2) *For any $a, b \in S, \theta_{\rho_a,\rho_b} = \rho_{a^*abb^*}$. Therefore, $\rho_a \circ \rho_b = \rho_a\rho_{a^*abb^*}\rho_b$.*

By the lemma above and Theorem 2.2 of [5], we can easily see the following lemma.

Lemma 3.2 *Define a mapping $\phi : S \rightarrow \mathcal{LI}_{S(\pi')}(\mathcal{M})$ by*

$$a\phi = \rho_a.$$

Then ϕ is a $*$-monomorphism.

Now, we have the main theorem.

Theorem 3.3 *A locally inverse $*$-semigroup can be embedded up to $*$-isomorphism in the π-symmetric locally inverse $*$-semigroup $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ on a π-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $\mathcal{M}$ determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$.*

If S is a generalized inverse $*$-semigroup, then a π-set $S(\pi'; \omega; \{\sigma_{e,f}\})$, constructed above, is a strong π-set. For, let $(e, f), (f, g) \in \omega$. Then there exist $h, k \in E(S)$ such that $e\mathcal{R}h\mathcal{L}f$ and $f\mathcal{R}k\mathcal{L}g$. Since S is a generalized inverse $*$-semigroup, $efg = eg \in E(S)$ and $e\mathcal{R}eg\mathcal{L}g$. In this case, it follows from [7] that $\sigma_{e,f}\sigma_{f,g} = \sigma_{e,g}$, and hence $S(\pi'; \omega; \{\sigma_{e,f}\})$ is a strong π-set. Then we have the following corollary.

Corollary 3.4 (Theorem 4.8 [7]). *A generalized inverse $*$-semigroup can be embedded up to $*$-isomorphism in $\mathcal{GI}_{X(\pi')}$ on a strong π-set $X(\pi'; \omega; \{\sigma_{e,f}\})$.*

4 Wreath products

Let S and T be locally inverse $*$-semigroups. By Theorem 3.3, T can be embedded in the π-symmetric locally inverse $*$-semigroup $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ on a π-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $\mathcal{M}$ determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$. In this case, we can consider T as a locally inverse $*$-subsemigroup of $\mathcal{LI}_{X(\pi')}(\mathcal{M})$, and so denote it by $T(X)$.

By ${}^XS$, denote the set of all mappings from the family $\mathbf{T}$ of π-single subsets of $X(\pi'; \omega; \{\sigma_{e,f}\})$ into S, and define a multiplication on ${}^XS$ by

$$d(\psi_1\psi_2) = d(\psi_1) \cap d(\psi_2),$$
$$x(\psi_1\psi_2) = (x\psi_1)(x\psi_2).$$

For any $\alpha \in \mathcal{LI}_{X(\pi')}$ and $\psi \in {}^XS$, let us define ${}^\alpha\psi (\in {}^XS)$ by

$$^\alpha\psi = \alpha\theta_{\alpha,\psi}\psi,$$

where $\theta_{\alpha,\psi} = \theta_{r(\alpha),d(\psi)} \in \mathcal{M}$.

Let $\psi \in {}^XS$ and $\alpha \in \mathcal{LI}_{X(\pi')}$ such that $d(\psi) = d(\alpha)$. Define a mapping ψ_α^* $(\in {}^XS)$ by

$$d(\psi_\alpha^*) = d(\alpha^{-1}),$$
$$x\psi_\alpha^* = (x\alpha^{-1}\theta_{\alpha^{-1},\psi}\psi)^*.$$

Since $r(\alpha^{-1}) = d(\alpha) = d(\psi)$, $\theta_{\alpha^{-1},\psi} = 1_{d(\alpha)}$ and hence $x\psi_\alpha^* = (x\alpha^{-1}\psi)^*$ for all $x \in d(\psi_\alpha^*)$.

Now, we define *the (right) wreath product* $S\mathrm{wr}T(X)$ of S and $T(X)$ as follows:

$$S\mathrm{wr}T(X) = \{(\psi,\alpha) \in {}^XS \times T(X) : d(\psi) = d(\alpha)\},$$
$$(\psi,\alpha)(\varphi,\beta) = (\psi\,{}^\alpha\varphi, \alpha \circ \beta),$$
$$(\psi,\alpha) = (\psi_\alpha^*, \alpha^{-1}).$$

Let $(\psi,\alpha), (\varphi,\beta) \in S\mathrm{wr}T(X)$. Then

$$\begin{aligned} x \in d(\psi\,{}^\alpha\varphi) &\iff x \in d(\psi) \text{ and } x \in d({}^\alpha\varphi) = d(\alpha\theta_{\alpha,\varphi}\varphi) \\ &\iff x \in d(\alpha), x\alpha \in d(\theta_{\alpha,\varphi}) \text{ and } x\alpha\theta_{\alpha,\varphi} \in d(\varphi) \\ &\iff x \in d(\alpha), x\alpha \in d(\theta_{\alpha,\beta}) \text{ and } x\alpha\theta_{\alpha,\beta} \in d(\beta) \\ &\iff x \in d(\alpha\theta_{\alpha,\beta}\beta) = d(\alpha \circ \beta). \end{aligned}$$

Then $(\psi,\alpha)(\varphi,\beta) = (\psi{}^\alpha\varphi) \in S\mathrm{wr}T(X)$, and hence $S\mathrm{wr}T(X)$ is closed under the multiplication. It immediately follows from the definition of ψ_α^* that $S\mathrm{wr}T(X)$ is closed under the unary operation $*$.

Theorem 4.1 *Let S and $T(X)$ be locally inverse $*$-semigroups. Then $S\mathrm{wr}T(X)$ is a locally inverse $*$-semigroup. Moreover, we have*

$$P(S\mathrm{wr}T(X)) = \{(\psi, 1_A) \in S\mathrm{wr}T(X) \colon A \in \mathbf{T} \textit{ and } r(\psi) \subseteq P(S)\},$$
$$E(S\mathrm{wr}T(X)) = \{(\psi,\alpha) \in S\mathrm{wr}T(X) \colon \alpha \in E(T(X)) \textit{ and } r(\psi) \subseteq E(S)\}.$$

Next, we shall consider wreath products of generalized inverse $*$-semigroups. Let S and $T(X)$ $(\subseteq \mathcal{GI}_{X(\pi';\omega;\{\sigma_{e,f}\})}(\mathcal{M}))$ be generalized inverse $*$-semigroups.

Lemma 4.2 *Let A, B, C be a π-single subsets of a strong π-set $X(\pi';\omega;\{\sigma_{e,f}\})$, and let $\psi \in {}^XS$ such that $d(\psi) = C$. Then, for any $x \in d(1_A \circ 1_B \circ 1_C)$, $x\,{}^{1_A \circ 1_B}\psi = x\,{}^{1_A}\psi$.*

By using the lemma above, we have the following theorem.

Theorem 4.3 *Let S and $T(X)$ $(\subseteq \mathcal{GI}_{X(\pi')}(\mathcal{M}))$ be generalized inverse $*$-semigroups, then $S\mathrm{wr}T(X)$ is a generalized inverse $*$-semigroup.*

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